The Electric Field of an Eccentric Dipole in a Homogeneous Spherical Conducting Medium

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The electrical position of the heart with reference to the electrodes used in studying its field is unknown. For reasons presented, it is more likely eccentric; hence, the equation defining the field of an eccentric dipole in a spherical medium might be useful for projected experimental studies and for better understanding of the way in which a given electrical position determines the electrode potentials. A method introduced by Helmholtz was used for deriving the desired equation. It is discussed since its concepts are of considerable importance to other electrocardiographic problems, too. The more simple mathematical example dealing with the centric dipole in the sphere is discussed. The equation for the field of the eccentric dipole is given and data based upon it are presented in numerical and map form. The Helmholtz equation for the field in the spherical conductor produced by two small spherical electrodes arbitrarily located is also presented and briefly discussed.

The assumptions upon which the equilateral triangle of Einthoven is based are: (1) that the electric field generated by the heart may be regarded as not significantly different from that of a current dipole at the center of a homogeneous spherical conductor, and (2) that this origin is equidistant from the apices of the triangle defined by the standard limb leads. It is, of course, obvious that these assumptions disregard the eccentric location of the human heart, which is, at least in the geometric sense, considerably nearer the left than the right shoulder, and much farther from the junction of the left leg with the trunk than from the right shoulder. It is also much closer to the anterior than to the posterior surface of the chest. The spatial relations suggest, therefore, that its field may resemble that of an eccentric dipole more closely than that of a centric dipole. For this reason it seemed to us that the equation which defines the field of an eccentric dipole in a spherical medium might prove useful in connection with a projected experimental study of the field established by connecting electrodes placed on the precordium of the living subject, or in the heart of a cadaver, to a current source.

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* Aided by a grant from the S. S. Kresge Foundation.
† Aided in part by a grant from United States Public Health Service.

So far as we know, this particular equation is not available in the literature, but equations which define fields similar to that under consideration are well known, and Helmholtz long ago developed a method by which all problems relating to the distribution of steady currents in a spherical conductor can be solved.

Helmholtz also gave the expression for the potential at any point in such a medium when the input electrodes are on its surface. We have employed this method to obtain the equation sought and this will be better understood if some account of the manner in which it was derived is given. We take this occasion, therefore, to call attention to the importance of the article cited for those who are concerned with bioelectric phenomena, and to mention some matters discussed therein which have a bearing on electrocardiographic problems not directly related to our present subject.

IMPORTANT PRINCIPLES BEARING ON THE DISTRIBUTION OF BIOELECTRIC CURRENTS

Five years hence, a century will have passed since Helmholtz, a young man, 32 years of age, who had already made important contributions to theoretic physics, had been the first to measure the speed of the nerve impulse, and who had invented the ophthalmoscope, became interested in the bioelectric investigations of his friend, Emil du Bois Reymond. The result of this interest was an article, published in 1853, on the distribution of electric currents in volume conductors, which came very close to
exhausting the subject. It seems strange that there are so few references to this article in the literature concerned with the electrocardiogram and related subjects. It is listed in the bibliography of the book by Kraus and Nicolai published in 1910, but there is, so far as we know, no mention of it in the writings of Waller, Einthoven, or Lewis.

We do not propose to review the contents of this article in detail, but merely to present without proof its more important theorems and to explain how we have made use of them for the purpose stated in the preceding paragraph. The fundamental principles stated and explained by Helmholtz may be considered under three headings:

1. The Principle of Reciprocity. We shall not try to state this theorem in the exact form which Helmholtz gave it, but only to give some idea of its scope and significance. Let there be given a volume conductor of arbitrary shape and either homogeneous or heterogeneous with respect to its electrical properties, and two pairs of electrodes arbitrarily located within it. If electrodes A and B of one pair are employed to pass a current i through the volume conductor, the resulting potential difference between C and D, the electrodes of the other pair, must be the same as the potential difference between A and B when a current of magnitude i enters the conductor by way of C and D. In other words, the result of this hypothetic experiment is not altered by interchanging the connections to the detector and the connections to the generator. A few experiments performed recently indicate that this principle holds when the volume conductor is the body of a living subject and the current is one alternating twenty-five times per second. We may conclude, then, that when a current of a given magnitude i flows out of the heart through one of the elements, a, of its surface and returns to the heart through another element, b, the resulting potential difference between two electrodes, one on each arm, is equal to the potential difference between a and b when a current i enters the body by way of the arm electrodes.

2. The Principle of Superposition. Let there be given a system of conductors within which electromotive forces are arbitrarily distributed. The currents and the potential differences due to all these electromotive forces acting together are the same as those obtained by adding algebraically all the currents and potential differences that would be produced by the individual electromotive forces, each acting independently. And furthermore, the components, parallel to three mutually perpendicular axes, of the current flowing at a given point when all the electromotive forces are acting simultaneously, are each equal to the algebraic sum of the corresponding components when each of the electromotive forces is acting independently.

Such experimental evidence as is pertinent indicates that this principle is applicable to the electromotive forces generated by the heart. It is known, for example, that when the auricles and ventricles are activated simultaneously the auricular and ventricular deflections are summed algebraically, and that algebraic addition of the dextrocardiogram and the levo-cardiogram produces a replica of the bicardiogram.

3. The Principle of the "Electromotive Surface." This might also be called the principle of the Helmholtz double layer. The "electromotive surface" or double layer is, in fact, a layer of dipoles so placed that their axes are perpendicular to the surface on which they lie. Across such a layer of dipoles there is a potential difference of 4πM where M is the electric moment per unit area of surface.

The theorem states that for every conductor A in the interior of which there are arbitrary electromotive forces there is a double layer, which, if located on the surface of A, will produce in a second conductor, B, when this is brought into contact with A, the same currents and potential differences as would be produced there by the internal electromotive forces of A acting alone under the same circumstances. The proof of this theorem depends upon the principle of superposition. The principle of the "electromotive surface," in the form in which it is applicable to networks was rediscovered by Thévenin after it was first stated by Helmholtz, and in most texts on electrical engineering it is called "Thévenin's theorem."
The hypothetic method of finding the double-layer specified by the theorem in a given case is described by Helmholtz somewhat as follows: Assume that conductor \( A \) with its internal electromotive forces is isolated from all others. Choose any element of its surface and measure the difference in potential between it and each of the other surface elements. On each of these other elements place a double layer so constituted that the potential of its outer surface with respect to that of its inner surface is the same as the potential of that element with respect to the potential of the reference element. The double layer obtained by this imaginary procedure is the one sought and we shall call it the specified double layer, and designate its potential function by the symbol \( P \). Now the sign of the potential function of a double layer can be reversed at every point by reversing the positions of the positive and negative poles of its component dipoles. The potential function of the double layer which is the inverse (of that) of the specified double layer may then be represented by the symbol, \(-P\). We shall call this double layer the inverse double layer. When it is substituted for the specified double layer all parts of the surface of conductor \( A \) will be at the same potential and no current will flow in the second conductor \( B \), containing no electromotive forces of its own, when it is brought into contact with \( A \) to form the system \( AB \). According to the principle of superposition we may say, then, that at all points of the conductor \( B \) the potential function \((-P)\) of the inverse double layer is equal in absolute magnitude and opposite in sign to the potential function \((W_1)\) of the interior electromotive forces of \( A \). We have then the equation,

\[
W_1 - P = 0, \text{ or } W_1 = P, \tag{1}
\]

which states that when the internal electromotive forces of \( A \) and the inverse double layer are acting together in the combined conductor \( AB \) there is no field in \( B \), and that the specified double layer acting alone upon \( AB \) would produce in \( B \) currents and potential differences identical with those that would be produced by the internal forces of \( A \) acting alone upon the same system.

Consider now conductor \( A \) before, and again after, it has been brought into contact with \( B \) to form the system \( AB \). Let \( W_0 \) be the potential function of the internal forces of \( A \) at any point of \( A \) before this conductor is brought into contact with \( B \), and \( W_1 \), the potential function of these internal forces at the same point after \( A \) has been brought into contact with \( B \). The inverse double layer prevented any flow of current in \( B \) when it was brought into contact with \( A \), and consequently, the effect of this double layer and the internal forces of \( A \) acting together upon the system \( AB \) produced in \( A \) the same field as the interior forces of \( A \) acting alone upon \( A \) before it was brought into contact with \( B \). Hence, we have the equation,

\[
W_0 = W_1 + P, \text{ or } W_1 = W_0 - P, \tag{2}
\]

which states that the currents and potential differences in \( A \) after it has been brought into contact with \( B \) to form the system \( AB \) are the algebraic sums of the currents and potential differences produced in \( A \), before it was brought into contact with \( B \), by its internal forces alone, and the currents and potential differences produced in \( A \), after it was brought into contact with \( B \), by the specified double layer acting alone.

It is clear that the only part of the double layer, specified or inverse, that is essential to the argument is that which lies on that part of the surface of \( A \) subsequently brought into contact with \( B \). Helmholtz pointed out that when the latter is a network of linear conductors which is brought into contact with \( A \) at only two points, \( P_1 \) and \( P_2 \), the volume conductor, so far as any observations on \( B \) are concerned, may be replaced by a resistance \( R \) and a voltage \( E \). The resistance \( R \) of \( A \) is measured between \( P_1 \) and \( P_2 \), and \( E \) is the potential difference between these two points before \( A \) is brought into contact with \( B \). If the electromotive forces of \( A \) are constant the inverse double layer in this case is equivalent to a battery with a voltage equal to the open-circuit potential difference between \( P_1 \) and \( P_2 \) and connected in such a way as to oppose this potential difference. The specified double layer is simulated by reversing (the contacts) of this battery.

In their recent book, Gardner and Barnes\(^4\)}
write in reference to this principle in the form
in which it was stated by Thévenin: "When
written, this theorem pertained to the direct-
current, steady-state behaviour of networks,
but it has been generalized since to alternating
currents and transients."

This principle has a very important bearing
upon many of the questions which arise in
studies concerned with the electrical field of
the heart. It makes it possible to predict what
kind of changes in this field would result if a
part of the heart's surface were short-circuited
or insulated from the neighboring tissues, or if
part or all of the body were immersed in a con-
ducting field. It shows that the voltages acting
in the standard limb leads, taken one at a time,
are always the open-circuit potential differ-
ences between the limb electrodes, even when
these leads are taken with a low-resistance in-
strument, such as a string galvanometer. It is
the principle upon which is founded Ein-
thoven's law that the algebraic sum of the
deflections in Leads I and III is equal to the
deflection in Lead II. Helmholtz discovered
the principle and confirmed it experimentally
before Einthoven was born by showing that
such a law held for a field generated in a lump
of carbon, but it is quite certain that Einthoven
was never aware of this earlier work. Neither
the principle nor the law is valid when the
circumstances are such that bringing the con-
ductor B into contact with A results in a flow
of current through a polarizable element and
consequently in a counter electromotive force.
Helmholtz made this observation in one of his
experiments in which the volume conductor
was a solution of copper sulfate in a glass
vessel.

**The Equation for the Field of an
Eccentric Dipole**

We shall now consider the method described
by Helmholtz for finding the potential function
of a known electromotive force in a spherical
homogeneous volume conductor, and its appli-
cation to the problem of obtaining an expression
which defines the field of an eccentric dipole
in such a medium. This method is founded on
the known equations for the potential at any
point outside or inside a spherical surface bear-
ing a layer of positive charges varying in num-
ber per unit area from point to point. The equa-
tions are:

\[ V_0 = \frac{1}{r} F(\theta, \phi, U), \quad U = \frac{R}{r} \]  \hspace{1cm} (3)

and

\[ V_i = \frac{1}{R} F(\theta, \phi, V), \quad V = \frac{r}{R}. \]  \hspace{1cm} (4)

The first of these equations gives the potential
at any outside point, and the second the po-
tential at any inside point, when \( R \) is the radius
of the spherical surface, \( r \) is the distance of the
outside point or inside point, as the case may
be, from the center of this surface, \( F \) is almost
any continuous function, and \( \theta \) and \( \phi \) are
the angles which define the direction from the origin
of the point at which the potential is to be
evaluated.

Helmholtz showed that by differentiating
Equations (3) and (4) with respect to \( R \), it is
possible to obtain the corresponding equations
applicable when the single layer of positive
charges is replaced by a layer of negative
charges on a spherical surface of radius
\( R - \Delta R \) and a layer of positive charges on a
spherical surface of radius \( R + \Delta R \). It is to be
understood that the new positive charges are
each equal in number and magnitude to the
original charges, that the two new spherical
surfaces are concentric with the original
surface, and that \( \Delta R \) is an infinitesimal. The double
layer thus obtained is equivalent to a layer of
dipoles. We then have the expressions:

\[ P_s = 2 \frac{dV_s}{dR} \Delta R = \frac{2}{r^2} F'(U) \Delta R, \]  \hspace{1cm} (5)

and

\[ P_i = 2 \frac{dV_i}{dR} \Delta R = - \frac{2}{R^3} \left[ F(V) + \frac{r}{R} F'(V) \right] \Delta R, \]  \hspace{1cm} (6)

in which

\[ F'(U) = \frac{dF(U)}{dU}, \]

and

\[ F''(V) = \frac{dF'(V)}{dV}. \]
Assume that the potential function \( W \) of a given electromotive force is known when the conductor in which it exists is infinite and homogeneous. The problem is to find the double layer, that, located on a spherical surface of given radius surrounding the site of the known electromotive force, will have the same potential function as this force at points outside the surface on which it lies. It is clear that if such a double layer can be found, the inverse double layer will cancel that part of the field of the given electromotive force which extends beyond the spherical surface. The currents and potential differences inside this surface will then be the algebraic sums of those determined by the potential function of the electromotive force and those determined by that of the inverse double layer. For points outside the spherical surface we have then the equation:

\[
W - P_0 = 0; \quad W = P_0 = \frac{2}{r^2} F'(U) \Delta R;
\]

Consequently,

\[
F(U) = \frac{1}{2 \Delta R} \int r^2 W \, dU + c,
\]

where \( c \) is an arbitrary constant. Having found \( F(U) \) we can find \( F(V) \) and finally \( S \), the potential inside the spherical surface, from the equations,

\[
S = W - P; \quad (7)
\]

and

\[
S = 2 \left[ \frac{1}{R^2} F(V) + \frac{1}{r^2} F'(U) + \frac{r}{R^4} F''(V) \right] \Delta R. \quad (8)
\]

As an illustration of the procedure involved, we may consider the simple case of a centric dipole in a sphere of radius \( R \). The potential function of a dipole in an infinite medium is given by the expression,

\[
V_p = M \frac{\cos \theta}{r^2}, \quad (9)
\]

where \( M \) is the moment of the dipole, \( r \) is the distance of the point \( p \) where the potential \( V_p \) is measured, and \( \theta \) the angle between the positive direction of the axis of the dipole and the radius vector to that point. We have

\[
W = P_0 = M \frac{\cos \theta}{r^2}, \quad F'(U) = M \frac{\cos \theta}{2 \Delta R},
\]

\[
F(U) = \frac{MU \cos \theta}{2 \Delta R}, \quad F'(V) = \frac{M}{2a R} \cos \theta, \quad (12)
\]

and

\[
F(V) = \frac{Mr \cos \theta}{2R \Delta R}. \quad (13)
\]

or

\[
S = M \cos \theta \left[ \frac{1}{r^2} + \frac{2r}{R} \right].
\]

which is correct.

In order to find the equation for the field of an eccentric dipole in a spherical medium the potential function of the dipole for an unrestricted medium must be expressed in a form different from that employed in the case of the centric dipole. As the origin of our coordinate system we choose the center of the spherical surface with reference to which the dipole is eccentric and is to become the boundary of the conducting medium. Let \( R \) be the radius of this surface, \( fR \) the distance of the dipole from the origin, \( \lambda_1, \mu_1, \nu_1 \) the direction cosines, with respect to the coordinate axes, of the radius vector to the dipole, \( r \), the distance of the point \( p \) from the origin, \( \lambda, \mu, \nu \) the direction cosines of the radius vector to this point, and \( \gamma \) the cosine of the angle between the radius vector to the dipole and the radius vector to \( p \). For \( \gamma \) we have

\[
\gamma = \lambda \lambda_1 + \mu \mu_1 + \nu \nu_1. \quad (14)
\]

The distance of the point \( p \) from the dipole is given by the expression,

\[
[r^2 + (fR)^2 - 2rfR \gamma]^{1/2}, \quad (15)
\]

and the direction cosines of the line from the dipole to \( p \) are

\[
\frac{r \lambda - fR \lambda_1}{[r^2 + (fR)^2 - 2rfR \gamma]^{1/2}}, \quad \frac{r \mu - fR \mu_1}{[r^2 + (fR)^2 - 2rfR \gamma]^{1/2}}, \quad \frac{r \nu - fR \nu_1}{[r^2 + (fR)^2 - 2rfR \gamma]^{1/2}}. \quad (16)
\]

If the direction cosines of the axis of the dipole are \( \psi, \psi', \phi \), the cosine of the angle
between this axis and the line from the dipole to \( p \) is the sum of the products formed by multiplying each of these direction cosines by the corresponding direction cosine of the three given in Equation (16). The resulting expression multiplied by the moment \( M \) of the dipole and divided by the square of the distance (Equation 15) of the point \( p \) from the dipole gives for the potential, \( V_p \), at \( p \) when the medium is unrestricted, the expression

\[
V_p = M \frac{\psi_x(r_{\lambda} - fR_{\lambda}) + \psi_y(r_{\mu} - fR_{\mu}) + \psi_z(r_{\nu} - fR_{\nu})}{r^2 + (fR)^2 - 2frR_r^{1/2}}.
\]

(17)

We shall not take up the space that would be needed for the details of the remainder of the calculations required to transform this equation into that for the spherical medium, since the principle and the method involved have already been fully explained. The final equation is

\[
V_p = M \psi_x \left\{ \frac{r_{\lambda} - fR_{\lambda}}{(r^2 + f^2)^{1/2}} \right. \\
+ \frac{r_{\lambda} - r_{\lambda}^2}{R^2 + r^2 - 2frR_r^{1/2}} \left. + \gamma \right) \\
+ \frac{(f - r_{\gamma}) \lambda - (f - r_{\gamma}) \lambda}{f\gamma^2(1 - r^2)(r^2 + r^2 - 2frR_r^{1/2})} \right. \\
\frac{\gamma \lambda - \lambda_1}{f\gamma^2(1 - r^2)^2 R^2 + r^2 - 2frR_r^{1/2}} + \frac{\gamma \lambda - \gamma_1}{f\gamma^2(1 - r^2)^2 R^2 + r^2 - 2frR_r^{1/2}}
\]

(18)

plus four terms of similar form in which \( \psi_\phi \), \( \mu_1 \), and \( \mu_2 \) replace \( \psi_\phi \), \( \lambda_1 \), and \( \lambda_2 \) and four more in which \( \psi_\phi \), \( \nu_1 \), and \( \nu_2 \) replace \( \psi_\phi \), \( \lambda_1 \), and \( \lambda_2 \).

When the point \( p \) at which the potential is to be evaluated is on the surface of the spherical conductor, \( r \) is equal to \( R \) and this equation reduces to a considerably less complicated one, i.e.,

\[
V_p = \frac{M \psi_x}{R} \left\{ \frac{2(\lambda - f\lambda)}{(1 + f^2 - 2f\gamma)^{1/2}} \right. \\
+ \frac{2(\gamma - f\gamma)}{f(1 - \gamma)^2 (1 + f^2 - 2f\gamma)^{1/2}} + \frac{\gamma \lambda - \lambda_1}{f(1 - \gamma)^2 (1 + f^2 - 2f\gamma)^{1/2}}
\]

(19)

plus six additional terms corresponding to the eight additional terms in Equation (18).

The equation corresponding to Equation (18) which defines the field in a homogeneous spherical conductor when a current \( J \) enters it by way of a small spherical electrode, \( S \), and leaves it by an identical electrode, \( S' \), is

\[
V_p = \frac{J}{4\pi} \int \log \rho \left[ \frac{1}{r} \left( \frac{1 - f_1 f_2}{1 - f_2 f_2} + (1 + f_1 f_2 f_2 - 2f_1 f_3 f_2)^{1/2} \right) \right. \\
+ \frac{1}{R} (f_1^2 + f_1^2 - 2f_1 f_3 f_2)^{1/2} \right. \\
\left. \left. + \frac{1}{R} (f_1^2 + f_1^2 - 2f_1 f_3 f_2)^{1/2} \right) \right] 
\]

(20)

In this expression, \( R \) is the radius of the spherical conductor, \( \sigma \) its specific conductivity, \( f_1R \) and \( f_2R \) the distances of the input electrodes \( S \) and \( S' \) from the center of the conductor, \( f_1R \) the distance of the point \( p \) from the center of the conductor, \( \gamma_1 \), the cosine of the angle between the radius vector to \( S \) and the radius vector to \( p \), and \( \gamma_2 \) the cosine of the angle between the radius vector to \( S' \) and the radius vector to \( p \). When \( f_1 \) and \( f_2 \) are each equal to 1, this equation reduces to that given by Helmholtz.

The last term in the second member of Equation (18), and the same term in Equation (19), represent the value of the integral in Equation (8) when \( U \) is equal to zero. No corresponding term occurs in the equation for the centric dipole, Equation (20), or the equation given by Helmholtz, for in these cases the integral vanishes when \( U \) becomes zero. In the case of the eccentric dipole this integral must be regarded as a definite integral with the upper limit \( U \) and the lower limit zero.

It will be observed that Equation (18) does not, as might be expected, reduce to the equation for the centric dipole when \( f \) is replaced by zero; for, when this is done, the last two terms of the second member appear to become infinite (note that \( \lambda_1 \) and \( \gamma \) are not defined when the dipole is centric) and their sum takes the indeterminate form zero divided by zero. By the proper methods of finding the limiting value of such indeterminate expressions, it can, however, be shown that as \( f \) approaches zero the sum of the twelve terms of the second member of Equation (18) approaches as a limit the second member of Equation (13).

The expressions which give the potential at the ends of that diameter of the spherical medium which passes through the eccentric dipole are of simple form. For the end of this diameter which is nearer the dipole, the po-
potential is given by the expression,

\[ V_p = \frac{M \cos \theta}{R^2} \left( 3 - \frac{f}{(1 - f)^2} \right) \]  

(21)

For the opposite end of this diameter, the corresponding expression is

\[ V_p = \frac{M \cos \theta}{R^2} \left( 3 + \frac{f}{(1 + f)^2} \right) \]  

(22)

In these last two equations, \( \theta \) is the angle between the axis of the dipole and the diameter specified.

**Table 1.—Potential values \((V_R, V_L, V_F)\) at the limb electrodes and their differences \((LII, LII, LIII)\) calculated with equation (19) for giving arbitrarily chosen eccentric positions \((a, b, c, d, e)\) of the dipole for a comparison with those due to the centric position \((f)\). In each dipole position the potentials are determined for both the horizontal \((H)\) and the vertical \((V)\) directions of the dipole axis. (See figure 1 for a map of these data in Burger triangles).**

<table>
<thead>
<tr>
<th>Eccentric Dipole</th>
<th>Position of Axis</th>
<th>( V_R )</th>
<th>( V_L )</th>
<th>( V_F )</th>
<th>( I )</th>
<th>( II )</th>
<th>( III )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (a) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td></td>
<td>7.64</td>
<td>3.82</td>
<td>-3.82</td>
</tr>
<tr>
<td>( b )</td>
<td>0.25</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1.86</td>
<td>1.86</td>
<td>1.86</td>
</tr>
<tr>
<td>( c )</td>
<td>0.25</td>
<td>0.866</td>
<td>-0.5</td>
<td>0</td>
<td>6.47</td>
<td>5.25</td>
<td>4.60</td>
</tr>
<tr>
<td>( d )</td>
<td>0.25</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3.74</td>
<td>3.32</td>
<td>4.81</td>
</tr>
<tr>
<td>( e )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6.40</td>
<td>6.40</td>
<td>6.40</td>
</tr>
<tr>
<td>( f )</td>
<td>0</td>
<td>Centric Dipole</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

An estimate of the error in surface potentials made by considering two small input electrodes equivalent to a doublet is given by

\[ V_p = \frac{3M \cos \theta}{R^2} \left( 1 - \frac{7a^2}{6R^2} \right) \]  

(23)

in which \( a \) is one half the distance between the centers of the electrodes. This equation is applicable to the case in which the input electrodes are equidistant from the center of a homogeneous spherical medium of radius \( R \).

The corresponding equation for the case in which the input electrodes are equidistant from an eccentric point is too complicated to be useful. The equation given is based on a similar one developed by Dr. Kenneth S. Cole.

**Illustration Examples**

Equation (19) may be used for the purpose of obtaining information as to the possible effects of the heart’s eccentric position upon the deflections in the standard or unipolar limb leads. It is, however, more particularly helpful in connection with the analysis of the results obtained in experiments in which an electric field is set up in the body by connecting two electrodes in contact with it to a current source.

In the discussion of electric fields that are to be compared with that of the heart it is desirable, because of the conventions that have been adopted in electrocardiography, to choose as coordinate axes the following lines:

- Horizontal or X-axis: A line extending from the origin toward the left limit of the cardiac field.
- Vertical or Y-axis: A line extending from the origin toward the inferior or caudal limit of the cardiac field.
- Sagittal or Z-axis: A line extending from the origin toward the posterior or dorsal limit of the cardiac field.

The results of some calculations based on Equation (19) are given in table 1. This table gives the value of \( f, \mu_1, \mu_2, \) and \( \eta \), assumed in each of five examples. The potentials calculated were those of the apices \( R, L, F \) of an equilateral triangle lying in the great circle of a spherical conductor of unit radius, in the XY plane, and in the same position as the
Einthoven triangle. The symbols \( V_R, V_L, \) and \( V_F, I, II, \) and III have the significance attached to them in discussions of the electrocardiogram. The potentials of the apices of the triangle and the differences in potential between them were computed for two positions of the dipole axis: the horizontal position, \( H, \) in which the axis is parallel to the X-axis; and the vertical position, \( V, \) in which it is parallel to the Y-axis. The moment of the dipole is taken as unity. The direction cosines given are with respect to the axes specified. In Example (a) the eccentric dipole is at the center of that side of the equilateral triangle which corresponds to Lead I. In Example (b) it is on the positive X-axis; in (c) it is on the line from the origin to the L apex; in (d) it is on the Y-axis below the origin; in (e) it is on the Z-axis at the posterior surface of the spherical conductor. For comparison, the data for a centric dipole (f) of unit moment are added at the end of the table.

In figure 1 the data contained in table 1 are presented in graphic form. The triangles shown there were constructed by the method devised by Burger and van Milaan which is based on the concept that the deflection in a given lead may be regarded as the scalar product of two vectors. One of these vectors represents the lead, and the other represents the electromotive force responsible for the field. In the cases under consideration, it is assumed that the moment of the dipole is equal to 1 and the second vector is, therefore, of unit length. Under the circumstances the lead vector is the hypotenuse of a right triangle. The altitude of this triangle is equal to the deflection in the given lead when the axis of the dipole is vertical, and the "base" is equal to this deflection when the axis of the dipole is horizontal. When data of the kind listed in table 1 are available, the simplest method of constructing the Burger triangle is to select some point as origin and locate its apices. To locate the \( F \) apex of the triangle (a), for example, it is necessary only to find on the positive Y-axis a point distant 1.55 units from the origin. To find the L apex required the location of a point distant 3.82 units from the positive X-axis and 0.31 units from the negative Y-axis.

The Burger triangle may be regarded as a kind of map which gives all the information concerning an electric field obtainable by meas-

\[ \text{Fig. 1.—Data in table 1 presented in graphic form.} \]
uring the potential differences between the three electrodes. In the hypothetic cases under consideration the true potentials of the electrodes are given for the two positions of the electrical axis. Consequently, the origin, 0, in the triangles of figure 1 represents a point in the field which is at zero potential. The lengths of the lines from this point to the apices of the triangle and from one apex to another are equal to the difference in potential between the two ends of that line when the axis of the dipole is parallel to it.

When the volume conductor is homogeneous and spherical, the dipole is centric, and the electrodes are on the surface of the conductor, the Burger triangle has exactly the same shape as the geometric triangle defined by the three lead electrodes. Under other circumstances, the shape of the former varies not only with the locations of the lead electrodes, but also with the shape, size, and nature of the volume conductor, and the site of its internal electromotive force.

Figure 1 and table 1 show that changes in the shape of the equilateral Burger triangle which depicts the field of a centric dipole in a spherical medium (f) produced by shifting the dipole to an eccentric position are in essence quite simple. When the dipole is moved toward the center of a given side of the triangle (a) that side becomes longer than the other two; when it is moved toward a given apex, Examples (c) and (d), the two sides of the triangle meeting at that apex become longer than the third side; when it is moved in a direction perpendicular to and away from the plane defined by the lead electrodes, Example (e), all of the sides of the triangle shorten to the same extent. Other displacements of the dipole may be considered combinations of these three. Triangles (a), (c), and (d) of figure 1 are isosceles. In (a) the horizontal side is longer; in (d) it is much shorter than the other two sides. In (c) the side cor-

responding to Lead II is shorter than those which correspond to Leads I and III. It may be pointed out that Burger triangles identical with these in shape could be obtained by shifting the lead electrodes instead of the dipole. If this were done, it would, however, be necessary in case (a) to locate these electrodes in such a way as to place the centric dipole outside the geometric triangle formed by the three leads.

Summary

Attention is called to the importance of the contribution made by Helmholtz to our knowledge of the distribution of electric currents, particularly bioelectric currents in volume conductors.

The equation which defines the potential function of an eccentric dipole in a homogeneous spherical medium is presented and discussed.

It is hoped that the equation will prove useful in connection with the analysis of the results of experiments in which an artificial electric field is established in the trunk by connecting electrodes placed on the precordium of a living subject, or in the heart of a cadaver, with a current source.

References

The Electric Field of an Eccentric Dipole in a Homogeneous Spherical Conducting Medium
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_Circulation_. 1950;1:84-92
doi: 10.1161/01.CIR.1.1.84

_Circulation_ is published by the American Heart Association, 7272 Greenville Avenue, Dallas, TX 75231
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Print ISSN: 0009-7322. Online ISSN: 1524-4539

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